J Math Chem (2014) 52:2459–2474
DOI 10.1007/s10910-014-0394-1

ORIGINAL PAPER

Analytic approximation to delayed convection dominated systems through transforms

Manju Sharma · Aditya Kaushik · Chenglin Li

Received: 1 April 2014 / Accepted: 17 July 2014 / Published online: 3 August 2014

Abstract In this paper we consider a family of singularly perturbed delay differential equation of convection diffusion type. When the perturbation parameter is very small, the solution of the problem exhibits layer behavior. In the layer region the solution changes rapidly, while away from this region the change in the solution is moderate. This simultaneous presence of two different scales phenomena makes the problem stiff. In this work, the problem is solved by applying a new Liouville–Green transform and the asymptotic solutions are obtained. Application to multi-point boundary value problem is also illustrated. Several test examples are taken into account so as to test the efficiency of the proposed method. The method presented is compared with other existing numerical or asymptotic methods. It is observed that the method presented is very easy to implement and is capable of reducing the size of calculations significantly while still maintaining high accuracy of the solution.

Keywords Asymptotic solution · Liouville Green transform · Layer behavior · Delay differential equation

The author greatfully acknowledges financial support from University Grants Commission (New Delhi, India) under Start-up Grant F.20-30(12)/2012(BSR).

A. Kaushik (✉)
University Institute of Engineering and Technology, Panjab University, Chandigarh 160014, UT, India
e-mail: akaushik@pu.ac.in

M. Sharma
Department of Mathematics, KVA DAV College, Karnal 132001, Haryana, India

C. Li
Department of Mathematics, Honghe University, Mengzi 661100, Yunnan, People's Republic of China

1 Introduction

We consider the Dirichlet boundary value problem

$$\begin{cases} \epsilon u'' + a(x)u'(x) + b(x)u(x-\delta) = 0 \quad \text{in } \Omega = [0,1], \\ u(x) = \phi(x) \text{ on } \ \Omega_0 \equiv -\delta \le x \le 0, \\ u(1) = \gamma, \end{cases} \tag{1.1}$$

where $0 < \epsilon \ll 1$ is a small parameter and δ is of $o(\epsilon)$. Further, it is assumed that $a(x), \phi(x) \in C^2[0,1]$, $|a(x)| \ge \theta > 0$ for all $x \in \Omega$, $b(x) \in C[0,1]$ and γ is a constant. This problem is the simple linear one dimensional model of convection diffusion problems with dominating convection term. Differential equations with a small parameter ϵ multiplying the highest order derivative terms are said to be singularly perturbed. Delay differential equation with small dissipating parameter occur frequently in engineering applications and in environmental sciences; for example, in fluid flow at high Reynolds number [1,2], advection dominated heat and mass transfer, semiconductor device models [3], theory of plates and shells [4], magneto-hydrodynamic flow [5], neuron variability [6–8] and in the study of travelling wave solutions [9]. Broad selection of such type of problems of the physics or engineering may be found in [3,10, 11].

In the case of singularly perturbed boundary value problem, for which accurate estimates of the diffusive fluxes are required [12], methods must be involved which approximate both the solution and the normalized fluxes accurately. Investigations of such methods have been sparse in the literature (see [1] for example).

In recent years many numerical methods; for example, finite differences [13–16], finite elements [17–19] and others [20–22] have been used to solve efficiently this type of problems (see [1,2] and references given therein), i.e., methods for which the associated error is bounded independently of ϵ. Among the different techniques used to find robust methods, the construction of specially adapted meshes is the most developed, since it permits to prove parameter uniform convergence of standard methods like central differences, simple upwind scheme or Galerkin methods, which fail on uniform meshes. Moreover, these meshes are also appropriate to integrate both two dimensional elliptic or time dependent problems (see [6,9,12,23,24] for example). Nevertheless, in general the methods achieve at most first order of uniform convergence. In this work, we study the second order differential difference equation (1.1) by applying a new Liouville Green transform and obtain the asymptotic solutions. As an application, we shall apply our results to a multi point boundary value problem. Moreover, we adopt following notational convention

$$\|g\|_\infty \equiv \|g\|_{\infty,\Omega} = max_x|g(x)|, \quad \|g\|_1 = \|g\|_{1,\Omega} = \int_0^1 |g(x)|dx$$
$$\|g\|_{\infty,k} \equiv \|g\|_{\infty,\bar{\Omega}_k} \text{ and } \|g\|_{1,k} \equiv \|g\|_{1,\Omega_k}, \quad k = 0, 1, 2.$$

2 The continuous problem and auxiliary results

In this section, we present some properties of the continuous problem. It is assumed that $a(x) \neq 0$ for all $x \in \Omega$. Immediately, two cases arise

$$a(x) \geq \theta > 0 \quad \text{or} \quad a(x) \leq -\theta < 0, \quad \text{for all} \quad x \in \Omega$$

where θ is some positive constant. We shall obtain the result for $a(x) \geq \theta > 0$, for $a(x) \leq -\theta < 0$ similar result may easily be obtained by transforming $x \to 1 - x$. For $a(x) \geq \theta > 0$ we have following estimate:

Lemma 2.1 *Let* $a(x),\ b(x) \in C(\bar{\Omega}),\ \phi(x) \in C(\Omega_0)$ *and*

$$\rho := \theta^{-1}\|b\|_{\infty,2}(1-\delta) < 1. \tag{2.1}$$

Then for the solution $u(x)$ *of the problem, the following estimates hold:*

$$\|u\|_{\infty} \leq C_0 \tag{2.2}$$

$$|u'(x)| \leq C\left(1 + \frac{1}{\epsilon}e^{-\frac{\theta x}{\epsilon}}\right), \quad 0 \leq x \leq 1 \tag{2.3}$$

where

$$C_0 = (|\phi(0)| + |\gamma| + \theta^{-1}\|b\|_{\infty,2}\|\phi\|_{1,0})(1-\rho)^{-1}.$$

Proof From the problem

$$u'(x) = u'(0)e^{-\frac{1}{\epsilon}\int_0^x a(\eta)d\eta} - \frac{1}{\epsilon}\int_0^x F(\xi)e^{-\frac{1}{\epsilon}\int_\xi^x a(\eta)d\eta}d\xi \tag{2.4}$$

with

$$F(x) = b(x)u(x-\delta).$$

Now integrating the Eq. (2.4) over $(0, x)$, we get

$$u(x) = \phi(0) + u'(0)\int_0^x e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau - \frac{1}{\epsilon}\int_0^x d\xi\, F(\xi)\int_\xi^x e^{-\frac{1}{\epsilon}\int_\xi^\tau a(\eta)d\eta}d\tau \tag{2.5}$$

Now using the condition $u(1) = \gamma$, we have

$$u'(0) = \frac{\gamma - \phi(0) + \frac{1}{\epsilon}\int_0^1 d\xi\, F(\xi)\int_\xi^1 e^{-\frac{1}{\epsilon}\int_\xi^\tau a(\eta)d\eta}d\tau}{\int_0^1 e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau} \tag{2.6}$$

Using Eqs. (2.5) and (2.6), we get

$$u(x) = \phi(0) + \left(\gamma - \phi(0) + \frac{1}{\epsilon}\int_0^1 d\xi\, F(\xi)\int_\xi^1 e^{-\frac{1}{\epsilon}\int_\xi^\tau a(\eta)d\eta}d\tau\right) \times \frac{\int_0^x e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau}{\int_0^1 e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau} - \frac{1}{\epsilon}\int_0^x d\xi\, F(\xi)\int_\xi^x e^{-\frac{1}{\epsilon}\int_\xi^\tau a(\eta)d\eta}d\tau. \quad (2.7)$$

Consider the Green's function

$$G(x,\xi) = \frac{1}{\epsilon}\int_\xi^1 e^{-\frac{1}{\epsilon}\int_\xi^\tau a(\eta)d\eta}d\tau\, \frac{\int_0^x e^{-\frac{1}{\epsilon}\int_0^s a(\eta)d\eta}ds}{\int_0^1 e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau} - \frac{1}{\epsilon}T_0(x-\xi)\int_\xi^x e^{-\frac{1}{\epsilon}\int_\xi^\tau a(\eta)d\eta}d\tau, \quad (2.8)$$

where

$$T_0(\lambda) = \begin{cases} 1, & \lambda \geq 0 \\ 0, & \lambda < 0 \end{cases}$$

The relation (2.7) can be written as

$$u(x) = \left(1 - \frac{\int_0^x e^{-\frac{1}{\epsilon}\int_0^s a(\eta)d\eta}ds}{\int_0^1 e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau}\right)\phi(0) + \frac{\int_0^x e^{-\frac{1}{\epsilon}\int_0^s a(\eta)d\eta}ds}{\int_0^1 e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau}\gamma + \int_0^1 G(x,\xi)F(\xi)d\xi. \quad (2.9)$$

Alternatively, the Green's function of the operator

$$Lu = -\epsilon u''(x) - a(x)u'(x), \quad 0 < x < 1,$$
$$u(0) = 0, \ u(1) = 0$$

can be expressed as

$$G(x,\xi) = \frac{1}{\epsilon v(\xi)}\begin{cases} \phi_1(\xi)\phi_2(x), & 0 \leq \xi \leq x \leq 1, \\ \phi_1(x)\phi_2(\xi), & 0 \leq x \leq \xi \leq 1, \end{cases} \quad (2.10)$$

where the functions ϕ_1 and ϕ_2 are the solutions of the following problems respectively

$$L\phi_1 = 0, \ \phi_1(0) = 0, \ \phi_1(1) = 1,$$
$$L\phi_2 = 0, \ \phi_2(0) = 0, \ \phi_2(1) = 1,$$

and

$$v(\xi) = \phi(\xi)/Q(1),$$
$$Q(x) = \int_0^x \phi(s)ds, \quad \phi(\xi) = e^{-\frac{1}{\epsilon}\int_0^\xi a(\tau)d\tau}.$$

Relation (2.10) implies $G(x, \xi) \geq 0$ and Eq. (2.8) shows that

$$\max_{x;\xi\in\bar{\Omega}} G(x, \xi) \leq \frac{1}{\epsilon}\left(\epsilon\theta^{-1}(1 - e^{-\frac{\theta}{\epsilon}(1-\xi)})\right)$$

Therefore $G(x, \xi) \leq \theta^{-1}$. This relation and the Eq. (2.9) gives

$$|u(x)| \leq |\phi(0)| + |\gamma| + \theta^{-1}\int_0^1 |b(\xi)u(\xi - \delta)|d\xi$$

Replacing the integral variables by $\xi = \delta + s$, we get

$$|u(x)| \leq |\phi(0)| + |\gamma| + \theta^{-1}\|b\|_{\infty,1}\|\phi\|_{1,0} + \theta^{-1}\|b\|_{\infty,2}(1 - \delta)\|u\|_\infty$$

which proves (2.2). Since

$$\int_0^1 e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau \geq \int_0^1 e^{-\frac{a^*\tau}{\epsilon}}d\tau \geq \frac{\epsilon}{a^*}\left(1 - e^{-a*}\right) \equiv c_0\epsilon$$

where $a^* = \|a\|_\infty$, and

$$\frac{1}{\epsilon}\int_0^1 d\xi|F(\xi)|\int_\xi^1 e^{-\frac{1}{\epsilon}\int_\xi^1 a(\eta)d\eta}d\tau \leq \theta^{-1}\|b\|_\infty\|\phi\|_{1,0} + \theta^{-1}\|b\|_\infty C_0(1 - \delta) \equiv C_1.$$

Moreover, relation (2.6) yields

$$\begin{aligned}|u'(0)| &\leq \frac{|\gamma| + |\phi(0)| + \frac{1}{\epsilon}\int_0^1 d\xi|F(\xi)|\int_\xi^1 e^{-\frac{1}{\epsilon}\int_\xi^\tau a(\eta)d\eta}d\tau}{\int_0^1 e^{-\frac{1}{\epsilon}\int_0^\tau a(\eta)d\eta}d\tau} \\ &\leq \frac{c_0^{-1}(|\gamma| + |\phi(0)| + C_1)}{\epsilon} \equiv \frac{C_2}{\epsilon}.\end{aligned}$$

Using the preceding in (2.4) and arguing the same the result (2.3) follows. □

Remark 2.1 It is assumed that $a(x) \neq 0, \quad \forall x \in \Omega$. However, it may be zero for finite set of points in Ω. In that case, the solution of the problem exhibits turning point behavior and Ω may be divided into finite number of subintervals. Then, Lemma (2.1) may be applied accordingly depending on the sign of $a(x)$ on that particular subinterval.

3 Solution methodology using Liouville–Green transform

An application of Taylor series in (1.1) yields

$$\begin{cases} \epsilon u''(x) + (a(x) - \delta b(x))u'(x) + b(x)u(x) = 0 \quad \text{in } \Omega = [0, 1], \\ u(0) = \phi(0) \\ u(1) = \gamma. \end{cases} \tag{3.1}$$

Define Liouville–Green transformation s, $\alpha(x)$ and $w(s)$ as follows

$$\begin{cases} s = \alpha(x) = \frac{1}{\epsilon}\int (a(x) - \delta b(x))dx, \\ \beta(x) = \alpha'(x) = \frac{1}{\epsilon}(a(x) - \delta b(x)), \\ w(s) = \beta(x)u(x). \end{cases} \tag{3.2}$$

From (3.2) it follows that

$$u'(x) = \frac{\alpha'(x)}{\beta(x)}\frac{dw}{ds} - \frac{\beta'(x)}{\beta^2(x)}w, \quad \text{and} \tag{3.3a}$$

$$u''(x) = \frac{\alpha'^2(x)}{\beta}\frac{d^2w}{ds^2} + \left(\frac{\alpha''(x)}{\beta} - 2\frac{\alpha'(x)\beta'(x)}{\beta^2(x)}\right)\frac{dw}{ds} - \left(\frac{\beta''(x)}{\beta^2(x)} - 2\frac{\beta'^2(x)}{\beta^3(x)}\right)w. \tag{3.3b}$$

Substitute (3.3) into (3.1), it gives

$$\frac{d^2w}{ds^2} + \left(\frac{\alpha''(x)}{\alpha'^2(x)} - 2\frac{\alpha'(x)\beta'(x)}{\beta(x)\alpha'^2(x)} + \frac{(a(x) - \delta b(x))\alpha'(x)}{\epsilon\alpha'^2(x)}\right)\frac{dw}{ds} -$$
$$\left(\frac{\beta''(x)}{\beta(x)\alpha'^2(x)} - 2\frac{\beta'^2(x)}{\alpha'^2(x)\beta^2(x)} + \frac{(a(x) - \delta b(x))\beta'(x)}{\epsilon\alpha'^2(x)\beta(x)} - \frac{b(x)}{\epsilon\alpha'^2(x)}\right)w = 0.$$

In view of (3.2), it reduces to

$$\frac{d^2w}{ds^2} + \frac{dw}{ds} = \epsilon F(x)\frac{dw}{ds} + \epsilon G(x, \epsilon)w(s) \tag{3.4}$$

where

$$F(x) = \frac{(a'(x) - \delta b'(x))}{(a(x) - \delta b(x))^2} \quad \text{and}$$

$$G(x, \epsilon) = \left(\frac{\epsilon(a''(x) - \delta b''(x))}{(a(x) - \delta b(x))^3} - 2\frac{\epsilon(a'(x) - \delta b'(x))^2}{(a(x) - \delta b(x))^4} + (a'(x) - \delta b'(x)) - b(x)\right).$$

Since $a(x) \in C^2[0, 1]$ and $b(x) \in C[0, 1]$, $F(x)$ and $G(x, \epsilon)$ are bounded on $[0, 1]$. Further, ϵ being sufficiently small

$$\epsilon F(x)\frac{dw}{ds} + \epsilon G(x, \epsilon)w(s) \to 0.$$

Therefore, (3.4) reduces to

$$\frac{d^2w}{ds^2} + \frac{dw}{ds} \approx 0, \tag{3.5}$$

and its solution is given by

$$w(s) = c_1 + c_2 \exp(-s) \tag{3.6}$$

where c_1 and c_2 are arbitrary constants. Combination of (3.2) and (3.6) gives

$$u(x) = \frac{\epsilon}{(a(x) - \delta b(x))}\left(c_1 + c_2 \exp\left(-\frac{1}{\epsilon}\int (a(x) - \delta b(x))dx\right)\right). \tag{3.7}$$

4 Application to multi-point boundary value problems

We illustrate further the idea of asymptotic solutions to second order multi point boundary value problem of type

$$\begin{cases} \epsilon u''(x) + (a(x) - \delta b(x))u'(x) + b(x)u(x) = 0 \quad \text{in } \Omega = [0, 1], \\ u(0) = \phi(0) = \phi, \\ u(1) - \sum_{i=1}^{n-2} \kappa_i u(\mu_i) = \gamma, \end{cases} \tag{4.1}$$

where $\phi,\ \gamma,\ \kappa_i,\ \mu_i$ $(i = 1, 2, \ldots, n-2)$ are finite constants such that

$$0 < \mu_1 < \mu_2 < \cdots < \mu_{n-2} < 1.$$

Suppose that

$$\Delta = \frac{\epsilon^2}{(a(0) - \delta b(0))} \left(\frac{e^{-\frac{1}{\epsilon}\int_0^1 (a(x)-\delta b(x))dx} - 1}{(a(1) - \delta b(1))} + \sum_{0}^{n-2} \kappa_i \left(\frac{1 - e^{-\frac{1}{\epsilon}\int_0^{\mu_i} (a(x)-\delta b(x))dx}}{(a(\mu_i) - \delta b(\mu_i))}\right)\right) \neq 0.$$

Asymptotic solution obtained in (3.7) when applied to the boundary conditions of (4.1) results into a system of two unknowns c_1^* and c_1^*, given by

$$c_1^* \frac{\epsilon}{(a(0) - \delta b(0))} + c_2^* \frac{\epsilon}{(a(0) - \delta b(0))} = \phi$$

and

$$c_1^* \epsilon \left(\frac{1}{(a(1) - \delta b(1))} - \sum_{i=1}^{n-2} \kappa_i \frac{1}{(a(\mu_i) - \delta b(\mu_i))}\right) + c_2^* \epsilon \left(\frac{e^{-\frac{1}{\epsilon}\int_0^1 (a(x)-\delta b(x))dx}}{(a(1) - \delta b(1))} - \sum_{i=1}^{n-2} \kappa_i \left(\frac{e^{-\frac{1}{\epsilon}\int_0^{\mu_i} (a(x)-\delta b(x))dx}}{(a(\mu_i) - \delta b(\mu_i))}\right)\right) = \gamma.$$

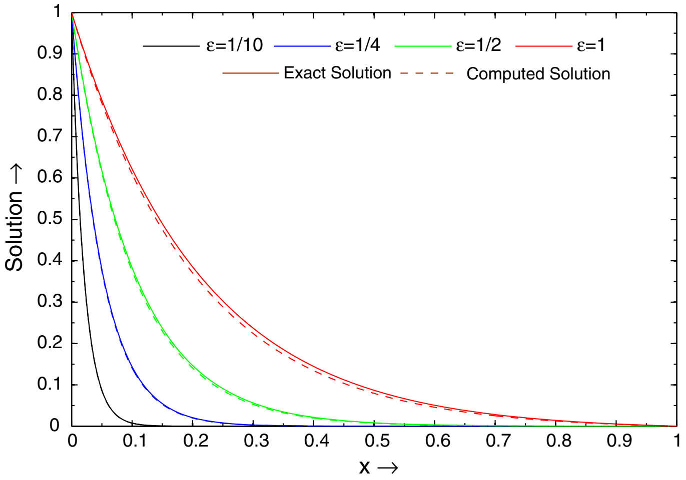

Fig. 1 Comparison of exact and computed solution for Example 5.1 when $\delta = 0.1$

It gives

$$c_1^* = \frac{\phi\epsilon\left(\frac{e^{-\frac{1}{\epsilon}\int_0^1 (a(x)-\delta b(x))dx}}{(a(1)-\delta b(1))} - \sum_{i=1}^{n-2}\kappa_i\left(\frac{e^{-\frac{1}{\epsilon}\int_0^{\mu_i} (a(x)-\delta b(x))dx}}{(a(\mu_i)-\delta b(\mu_i))}\right)\right) - \frac{\epsilon\gamma}{a(0)-\delta b(0)}}{\Delta} \tag{4.2}$$

and

$$c_2^* = \frac{\phi\epsilon\left(\frac{1}{(a(1)-\delta b(1))} - \sum_{i=1}^{n-2}\kappa_i\frac{1}{(a(\mu_i)-\delta b(\mu_i))}\right) - \frac{\epsilon\gamma}{a(0)-\delta b(0)}}{\Delta}. \tag{4.3}$$

Consequently, asymptotic solution of (4.1) reads as

$$u(x) \approx \frac{\epsilon}{a(x)-\delta b(x)}\left(c_1^* + c_2^* \exp(-\frac{1}{\epsilon}\int_0^x (a(t)-\delta b(t))dt)\right), \tag{4.4}$$

where c_1^* and c_2^* are constants given by the Eqs. (4.2) and (4.3).

5 Numerical illustration

In this section, several numerical examples are considered and solved using method presented in this paper. The exact solution of the boundary value problem (3.1) with constant coefficients (i.e., $a(x) = a$ and $b(x) = b$) reads

$$y(x) = \frac{\gamma - \phi\exp(m_2)}{\exp(m_1) - \exp(m_2)}\exp(m_1 x) + \frac{\phi\exp(m_1) - \gamma}{\exp(m_1) - \exp(m_2)}\exp(m_2 x) \tag{5.1}$$

Table 1 Numerical computations for Example 5.1 with $\epsilon = 2^{-2}$ and $\delta = 0.1$

x	Exact solution	Computed solution	Pointwise Error
0.1	0.14379381821116	0.14085841827929	0.00293539993187
0.2	0.02067665891105	0.01984109173050	0.00083556718055
0.3	0.00297317208765	0.00279478220875	0.00017838987890
0.4	0.00042752012898	0.00039366596699	0.00003385416200
0.5	0.00006147117862	0.00005544852472	0.00000602265390
0.6	0.00000883567395	0.00000780774988	0.00000102792407
0.7	0.00000126708504	0.00000109714556	0.00000016993948
0.8	0.00000017883874	0.00000015190043	0.00000002693831
0.9	0.00000002242424	0.00000001875470	0.00000000366954

Table 2 Numerical computations for Example 5.1 with $\epsilon = 2^{-3}$ and $\delta = 0.1$

x	Exact solution	Computed solution	Pointwise error
0.1	0.02025234949766	0.01984109474437	0.41125475328994E-3
0.2	0.00041015766018	0.00039366904066	0.01648861952029E−3
0.3	0.00000830665628	0.00000781082473	0.00049583154944E−3
0.4	0.00000016822931	0.00000015497531	0.00001325399262E−3
0.5	0.00000000340704	0.00000000307488	0.00000033215881E−3
0.6	0.00000000006900	0.00000000006101	0.00000000799154E−3
0.7	0.00000000000140	0.00000000000121	0.00000000018693E−3
0.8	0.00000000000003	0.00000000000002	0.00000000000427E−3
0.9	0.00000000000000	0.00000000000000	0.00000000000008E−3

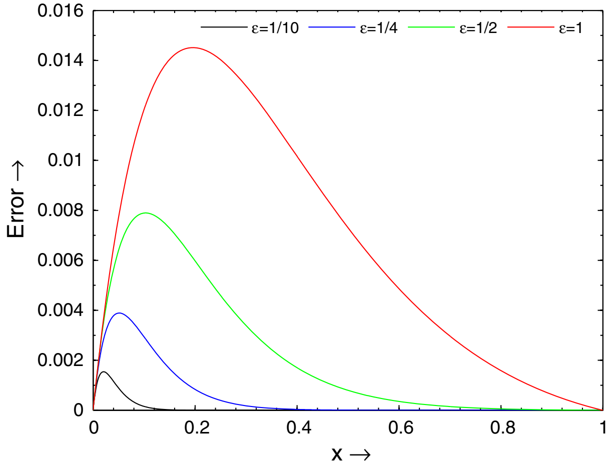

Fig. 2 Error plot for Example 5.1 for different values of ϵ when $\delta = 0.1$

Table 3 Numerical computations for Example 5.1 with $\epsilon = 2^{-1}$ and different values of δ

x	$\delta = 0.1$			$\delta = 0.01$		
	Exact solution	Computed solution	Pointwise error	Exact solution	Computed solution	Pointwise error
0.1	0.38316901966795	0.37527645693173	0.00789256273622	0.37619322771992	0.36858669108951	0.00760653663041
0.2	0.14678858824437	0.14081077750444	0.00597781073993	0.14149594163992	0.13583768303103	0.00565825860890
0.3	0.05620408542375	0.05281320571565	0.00339087970810	0.05319536901943	0.05004268950186	0.00315267951757
0.4	0.02149135887712	0.01978674035134	0.00170461852578	0.01997439332007	0.01841728763196	0.00155710568811
0.5	0.00818974850801	0.00739154134428	0.00079820716373	0.00747629063520	0.00675966051071	0.00071663012448
0.6	0.00309326860089	0.00273948558446	0.00035378301643	0.00277484840498	0.00246247337434	0.00031237503064
0.7	0.00114113246321	0.00099351742533	0.00014761503788	0.00100676928370	0.00087846171459	0.00012830756910
0.8	0.00039394993971	0.00033823619696	0.00005571374275	0.00034230937700	0.00029456977352	0.00004773960348
0.9	0.00010850990943	0.00009230187909	0.00001620803035	0.00009305472714	0.00007933789896	0.00001371682818

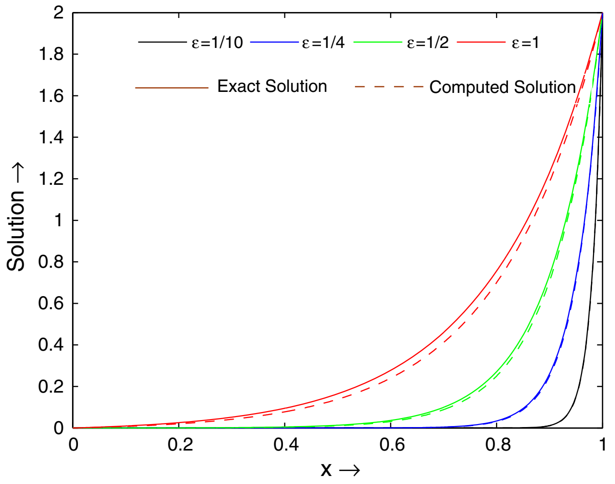

Fig. 3 Comparison of exact and computed solution for Example 5.2 when $\delta = 0.1$

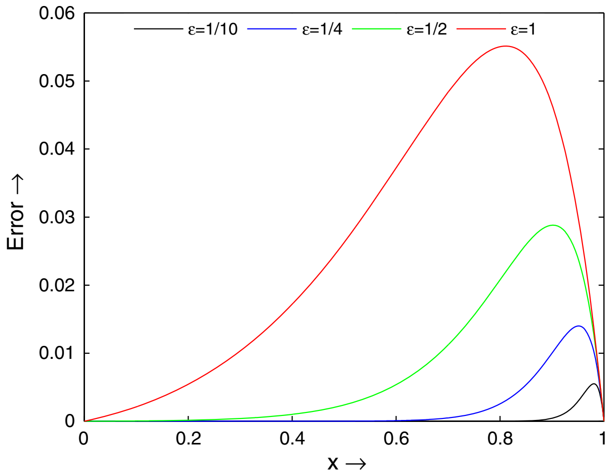

Fig. 4 Error plot for Example 5.2 for different values of ϵ when $\delta = 0.1$

where

$$m_1 = \frac{-(a-b\delta)+\sqrt{(a-b\delta)^2-4\varepsilon b}}{2\varepsilon} \quad \text{and} \quad m_2 = \frac{-(a-b\delta)-\sqrt{(a-b\delta)^2-4\varepsilon b}}{2\varepsilon}.$$

However asymptotic solution (3.7) subject to boundary condition $u(0) = \phi$ and $u(1) = \gamma$ reads

$$u(x) = \phi - \frac{\phi-\gamma}{1-\exp(-\frac{a-\delta b}{\epsilon})} + \frac{\phi-\gamma}{1-\exp(-\frac{a-\delta b}{\epsilon})} \exp\left(-\frac{a-\delta b}{\epsilon}x\right). \tag{5.2}$$

Table 4 Numerical computations for Example 5.2 with $\epsilon = 0.1$ and $\delta = 0.1$

x	Exact solution	Computed solution	Pointwise error
0.1	0.00000000000000	0.00000000000000	0.00000000000E−3
0.2	0.00000000000000	0.00000000000000	0.00000000000E−3
0.3	0.00000000000000	0.00000000000000	0.00000000000E−3
0.4	0.00000000000007	0.00000000000006	0.00000000001E−3
0.5	0.00000000001240	0.00000000001022	0.00000000218E−3
0.6	0.00000000216282	0.00000000185227	0.00000031055E−3
0.7	0.00000037715790	0.00000033576551	0.00004139239E−3
0.8	0.00006576964098	0.00006086496602	0.00490467496E−3
0.9	0.01146905758813	0.01103312884152	0.43592874661E−3

Table 5 Numerical computations for Example 5.2 with $\epsilon = 2^{-2}$ and $\delta = 0.1$

x	Exact solution	Computed solution	Pointwise error
0.1	0.00000001824795	0.00000001297418	0.00000001297418
0.2	0.00000015942757	0.00000011682562	0.00000011682562
0.3	0.00000124681219	0.00000094810121	0.00000094810121
0.4	0.00000961694147	0.00000760202085	0.00000760202085
0.5	0.00007404065246	0.00006086311380	0.00006086311380
0.6	0.00056989545628	0.00048718987663	0.00048718987663
0.7	0.00438637260010	0.00389970919703	0.00389970919703
0.8	0.03376088858255	0.03121511401660	0.03121511401660
0.9	0.25984953872717	0.24986042277630	0.24986042277630

Example 5.1 Consider (1.1) with $a = 5$ and $b = 1$ on Ω with boundary conditions $\phi = 1$ and $\gamma = 0$.

Example 5.2 Consider (1.1) with $a = -5$ and $b = 2$ on Ω with boundary conditions $\phi = 0$ and $\gamma = 2$.

Example 5.3 Consider next (1.1) with variable coefficient [25] $a = 1 - x/2$ and $b = -1/2$ on Ω with boundary conditions $\phi = 0$ and $\gamma = 1$. The asymptotic solution of the problem given by (3.7) reads

$$u(x) = \frac{1}{(2-x)(\exp(-3/4\epsilon) - 1)} \left(-1 + \exp \frac{-(x - x^2/4)}{\epsilon} \right).$$

We use uniformly valid solution $u(x) = 1/(2-x) - \exp(-(x - x^2/4)/\epsilon)$ obtained by Nayfeh [26] for comparision.

In case the perturbation parameter tends to zero the solution of the problem exhibits boundary layer behavior depending on the sign of the convection term (Figs. 1, 3, 5).

Table 6 Numerical computations for Example 5.2 with $\epsilon = 2^{-3}$ and different values of δ

x	$\delta = 0.1$			$\delta = 0.01$		
	Exact solution	Computed solution	Pointwise error	Exact solution	Computed solution	Pointwise error
0.1	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000	0.00000000000000
0.2	0.00000000000001	0.00000000000001	0.00000000000000	0.00000000000003	0.00000000000002	0.00000000000001
0.3	0.00000000000059	0.00000000000045	0.00000000000014	0.00000000000164	0.00000000000124	0.00000000000040
0.4	0.00000000003649	0.00000000002891	0.00000000000758	0.00000000008732	0.00000000006859	0.00000000001873
0.5	0.00000000224910	0.00000000185227	0.00000000039683	0.00000000465356	0.00000000380537	0.00000000084819
0.6	0.00000013861613	0.00000011867789	0.00000001993824	0.00000024799007	0.00000021111711	0.00000003687295
0.7	0.00000854314617	0.00000760387312	0.00000093927305	0.00001321548821	0.00001171251278	0.00000150297543
0.8	0.00052652850751	0.00048719172845	0.00003933677906	0.00070425856858	0.00064979552633	0.00005446304225
0.9	0.03245083997407	0.03121511583997	0.00123572413410	0.03753021632167	0.03604984122934	0.00148037509233

Table 7 Comparison of computed solution with exact solution and solution obtained in [27–29] for Example 5.3 when $\epsilon = 10^{-2}$

x	Exact solution	Computed solution	Andargie et al. [27]	Chawla [28,29]
0.1	0.5262867	0.5262851	0.4087878	0.4080370
0.2	0.5555555	0.5555555	0.4416590	0.4407397
0.3	0.5882353	0.5882353	0.4788066	0.4778685
0.4	0.6250000	0.6250000	0.5215091	0.5205625
0.5	0.6666667	0.6666667	0.5709993	0.5700607
0.6	0.7142857	0.7142857	0.6288814	0.6279783
0.7	0.7692308	0.7692308	0.6972787	0.6964532
0.8	0.8333333	0.8333333	0.7790488	0.7783682
0.9	0.9090909	0.9090909	0.8781180	0.8776891

Table 8 Comparison of computed solution with exact solution and solution obtained in [30] for Example 5.3 when $\epsilon = 10^{-3}$

x	Exact solution	Computed solution	Andargie et al. [27]	Chawla [28,29]	Andargie et al. [30]
0.1	0.5263157	0.5263157	0.4021985	0.3958319	0.5263206
0.2	0.5555555	0.5555555	0.4345494	0.4292463	0.5555605
0.3	0.5882352	0.5882352	0.4715418	0.4661086	0.5882403
0.4	0.6250000	0.6250000	0.5141674	0.5086559	0.6250049
0.5	0.6666666	0.6666666	0.5637082	0.5582095	0.6666715
0.6	0.7142857	0.7142857	0.6218457	0.6165099	0.7142904
0.7	0.7692307	0.7692307	0.6908247	0.6858976	0.7692348
0.8	0.8333333	0.8333333	0.7737027	0.7695873	0.8333368
0.9	0.9090909	0.9090909	0.8747349	0.8721039	0.9090930

For small values of ϵ, a comparision of exact and computed solution is made for Example 5.1 and Example 5.2 in the form of Tables 1, 2 and 4, 5 respectively. It is observed that, not only the perturbation parameter adds the boundary layer character to the problem but the small delay present in the reaction term also results into steep gradients (Figs. 2, 4). A comparision of exact solution with the computed solution in made for Example 5.1 and Example 5.2 for different value of delay argument. The results so obtained are tabulated in the form of Tables 3 and 6 respectively. Further, a variable coefficient problem is taken into account in the form of Example 5.3). The solution so obtained is compared with the available exact solution and solution obtained by other researchers [27,28,30]. The comparative results obtained are tabulated in Tables 7 and 8 and graphical illustration can be seen from Fig. 5.

It is observed that the numerical scheme is robust with respect to the perturbation parameter and that the scheme yields results much better than the existing numerical schemes like; fourth order tridiagonal finite difference methods (FDM) [28,29], fitted fourth order tridiagonal FDM [27] and exponential fitted FDM [30]. In case of Example

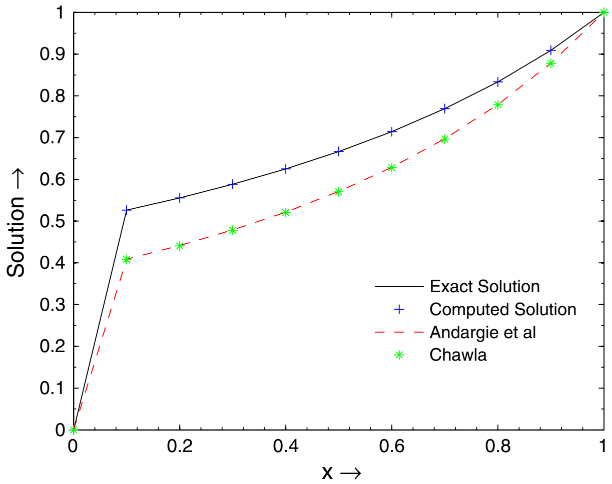

Fig. 5 Comparison of computed solution with exact solution and solution obtained in [27–29] for Example 5.3 when $\epsilon = 10^{-2}$

5.3, it is observed that the scheme not only yields accurate solution but the solution so obtained is comparable with the pure asymptotic solutions [25,26].

The method presented here is very easy to implement and with a little modification can easily be extended to even more general situations. In case of nonlinear problems, we make use of quasi-linearization technique so as to linearize the problem around a nominal solution. This is followed by the solution of the linear version of the same [21]. However, we will not go further into this as the issue is left for our further study. In summary, it can be said that the theoretical estimates are supported by the numerical results. Moreover, the comparison of the method presented here with different established numerical techniques shows that the method presented here is very easy to implement and with a little modification can easily be extended to even more general situations.

References

1. H.G. Roos, M. Stynes, L. Tobiska, *Numerical Methods for Singularly Perturbed Differential Equations* (Springer, Berlin, 1996)
2. J.J.H. Miller, E. ORiordan, G.I. Shishkin, *Fitted Numerical Methods for Singular Perturbation Problems, Error Estimates in the Maximum Norm for Linear Problems in One and Two Dimensions* (World Scientific, Singapore, 1996)
3. R.E. O'Malley, *Singular Perturbation Methods for Ordinary Differential Equations, Applied Mathematical Sciences 89* (Springer, Berlin, 1990)
4. J.K. Knowles, R.E. Messick, On a class of singular perturbation problems. J. Math. Anal. Appl. **9**, 42–58 (1964)
5. R.R. Gold, Magnetohydrodynamic pipe flow. Part I. J. Fluid Mech. **13**, 505–512 (1962)
6. A. Kaushik, Singular perturbation analysis of bistable differential equation arising in the nerve pulse propagation. Nonlinear Anal. **09**, 2106–2127 (2008)
7. A. Kaushik, M.D. Sharma, Numerical analysis of a mathematical model for propagation of an electrical pulse in a neuron. Numer. Methods Partial Differ. Equ. **27**, 1–18 (2008)

8. M.K. Kadalbajoo, K.K. Sharma, Numerical treatment of a mathematical model arising from a model of neuronal variability. J. Math. Anal. Appl. **307**, 606–627 (2005)
9. A. Kaushik, Nonstandard perturbation approximation and traveling wave solutions of non-linear reaction diffusion equations. Numer. Methods Partial Differ. Equ. **24**, 217–238 (2008)
10. R.E. O'Malley, *Introduction to Singular Perturbations* (Academic Press, New York, 1974)
11. J.J.H. Miller, *Singular Perturbation Problems in Chemical Physics, Analytic and Computational Methods XCVII* (Wiley, New York, 1997)
12. T. Linβ, An upwind difference scheme on a novel Shishkin type mesh for a linear convection diffusion problem. J. Comput. Appl. Math. **110**, 93–104 (1999)
13. M.K. Kadalbajoo, K.K. Sharma, Numerical analysis of singularly perturbed delay differential equations with layer behavior. Appl. Math. Comput. **157**, 11–28 (2004)
14. M.K. Kadalbajoo, K.K. Sharma, Numerical analysis of boundary-value problems for singularly perturbed differential-difference equations: small shifts of mixed type with rapid oscillations. Commun. Numer. Methods in Eng. **20**, 167–182 (2004)
15. A. Kaushik, M. Sharma, Convergence analysis of weighted difference approximations on piecewise uniform grids to a singularly perturbed functional differential equations. J. Optim. Theory Appl. **155**, 252–272 (2012)
16. A. Kaushik, M. Sharma, A Robust numerical approach for singularly perturbed time delayed parabolic partial differential equations. Comput. Math. Model. **23**, 96–106 (2012)
17. V. John, J. Maubach, L. Tobiska, Nonconforming streamline diffusion finite element methods for convection diffusion problems. Numer. Math. **78**, 165–188 (1997)
18. F. Schieweck, L. Tobiska, A nonconforming finite element method of upstream type applied to the stationary Navier–Stokes equation. RAIRO Modél. Math. Anal. Numér. **23**, 627–647 (1989)
19. D. Kay, D. Silvester, The reliability of local error estimators for convection diffusion equations. IMA J. Numer. Anal. **21**, 107–122 (2001)
20. A. Kaushik, V. Kumar, M. Sharma, Analysis of factorization method for elliptic differential equation. Comput. Math. Model. **22**, 98–110 (2011)
21. A. Kaushik, V. Kumar, A.K. Vashishth, An efficient mixed asymptotic-numerical scheme for singularly perturbed convection diffusion problems. Appl. Math. Comput. **118**, 8645–8658 (2012)
22. A. Kaushik, Iterative analytic approximation to nonlinear convection dominated systems. Comput. Phys. Commun. **184**, 2061–2069 (2013)
23. T. Linβ, M. Stynes, A hybrid difference scheme on a Shishkin mesh for linear convection diffusion problems, APNUM. **31**, 255–270 (1999)
24. A. Kaushik, K.K. Sharma, M. Sharma, A parameter uniform difference scheme for parabolic partial differential equation with a retarded argument. Appl. Math. Model. **34**, 4232–4242 (2010)
25. J. Kevorkian, J.D. Cole, *Perturbation Methods in Applied Mathematics* (Springer, New York, 1981)
26. A.H. Nayfeh, *Perturbation Methods* (Wiley, New York, 1979)
27. A. Andargie, Y.N. Reddy, Fitted fourth-order tridiagonal finite difference method for singular perturbation problems. Appl. Math. Comput. **192**, 90–100 (2007)
28. M.M. Chawla, A fourth-order tridiagonal finite difference method for general non-linear two-point boundary value problems with mixed boundary conditions. J. Inst. Maths Appl. **21**, 83–93 (1978)
29. M.M. Chawla, An efficient finite difference method for two-point boundary value problems. Neural Parallel Sci. Comput. **4**, 384–396 (1996)
30. A. Andargie, Y.N. Reddy, An exponentially fitted special second-order finite difference method for solving singular perturbation problems. Appl. Math. Comput. **190**, 1767–1782 (2007)